Problem 1

Two fish swim in a pond. One is a piranha and the other is either a clownfish with probability 0.5 or another piranha with probability 0.5.

1. A fisherman takes a fish from the pond. What is the probability that it is Nemo?
2. The fisherman observes it is a piranha. What is the probability that the other fish is Nemo?
3. Write a probabilistic program (over the simple probabilistic language) that encodes subproblem 2.
4. Show the probabilistic semantics of the program.
Solution (1)

Define a random variable $t$ for the type of the fish.  
The probability is  

$$P(t = \text{Nemo}) = P(\text{catches fish 1}) \times P(t = \text{Nemo}|\text{fish caught is 1}) + \  
P(\text{catches fish 2}) \times P(t = \text{Nemo}|\text{fish caught is 2}) = \  
0.5 \times 0 + 0.5 \times 0.5 = 0.25 \  $$
Solution (2)

Define a random variable \( o \) for the type of the other fish.

Using Bayes rule:

\[
\begin{align*}
P(o = Nemo | t = Piranha) &= \frac{P(t = Piranha | o = Nemo) \cdot P(o = Nemo)}{P(t = Piranha)} \\
&= \frac{1 \cdot (0.5 \cdot 0.5)}{1 - 0.25} \\
&= \frac{0.25}{0.75} = \frac{1}{3}
\end{align*}
\]
Solution (3)

\[ x_1 := 0; \]
\[ x_2 := \text{Bern}(1/2); \]
\[ (t, o) := (0,0); \]
\[ \text{if Bern}(1/2) \]
\[ \text{then } t = x_1; o = x_2; \]
\[ \text{else } t = x_2; o = x_1; \]
\[ \text{observe}(t = 1); \]
\[ \text{return } o; \]
Solution (4)

The state consists of the four (random) variables: $x_1, x_2, t, o$

State also contains the probability of a statement to be in this state

The initial state:

$$((x_1: \perp, x_2: \perp, t: \perp, o: \perp), 1)$$
Solution (4)

\[(x_1 \downarrow, x_2 \downarrow, t \downarrow, o \downarrow), 1\]

\[x_1 := 0;\]
\[(x_1 : 0, x_2 : \downarrow, t : \downarrow, o : \downarrow), 1\]

\[x_2 := \text{Bern}(1/2);\]
\[(x_1 : 0, x_2 : 0, t : \downarrow, o : \downarrow), 0.5\]  
\[(x_1 : 0, x_2 : 1, t : \downarrow, o : \downarrow), 0.5\]

\[(t, o) := (0,0);\]
\[(x_1 : 0, x_2 : 0, t : 0, o : 0), 0.5\]  
\[(x_1 : 0, x_2 : 1, t : 0, o : 0), 0.5\]

if Bern(1/2)
\[t = x_1; o = x_2;\]
\[(x_1 : 0, x_2 : 0, t : 0, o : 0), 0.25\]  
\[(x_1 : 0, x_2 : 1, t : 0, o : 1), 0.25\]

else
\[t = x_2; o = x_1;\]
\[(x_1 : 0, x_2 : 0, t : 0, o : 0), 0.25\]  
\[(x_1 : 0, x_2 : 0, t : 1, o : 0), 0.25\]

observe(t = 0);
\[(x_1 : 0, x_2 : 0, t : 0, o : 0), 0.25\]  
\[(x_1 : 0, x_2 : 0, t : 0, o : 1), 0.25\]

return o;
\[(x_1 : 0, x_2 : 0, t : 0, o : 0), 0.25\]  
\[(x_1 : 0, x_2 : 1, t : 0, o : 0), 0.25\]
Problem 2: The Monty Hall Problem

There are three doors. Behind two of them there are goats, behind the other one there is a car. Monty Hall tells you to pick a door. If you pick the one with the car, it is yours. You pick door 1. Now Monty Hall has to pick a different door that behind which there is a goat. He picks door 3. Now he offers you to switch to door 2.

1. Determine whether you should keep door 1 or switch to door 2. What is the winning probability in each of the cases?

2. Write a probabilistic program (over the simple probabilistic language) that encodes this problem.
Solution (1)

Define random variables:
• $d_1, d_2, d_3$ encoding that there is a car behind doors 1, 2, and 3
• $p$ that encodes the door you picked
  • $p$ and $d_i$ are independent
• $m$ that encodes the door Monty picked

By the game rules, Monty Hall has to pick a door different from yours, and show a goat. Given Monty Hall’s choice, we can infer:

$P(m = 3|d_3, p = 1) = 0$ (no car behind door 3, since he picked it)
$P(m = 3|d_2, p = 1) = 1$ (if the car is behind door 2, he has to pick door 3)
$P(m = 3|d_1, p = 1) = 0.5$ (if the car is behind door 1, he can pick door 2 or 3)
Solution (1)

The probability that behind door 1 there is a car is given by:

\[
P(d_1 = \text{Car} | m = 3, p = 1) = \frac{P(m = 3 | d_1 = \text{Car}, p = 1) \cdot P(d_1 = \text{Car} \cap p = 1)}{P(m = 3 | d_1 = \text{Car}, p = 1) \cdot P(d_1 = \text{Car} \cap p = 1)} =
\]

\[
P(m = 3 | d_1, p = 1) \cdot P(d_1 \cap p = 1) + P(m = 3 | d_2, p = 1) \cdot P(d_2 \cap p = 1) + P(m = 3 | d_3, p = 1) \cdot P(d_3 \cap p = 1) =
\]

\[
\frac{0.5 \cdot P(d_1 = \text{Car} \cap p = 1)}{0.5 \cdot P(d_1 \cap p = 1) + 1 \cdot P(d_2 \cap p = 1)} = \frac{0.5}{1.5} = \frac{1}{3}
\]

\[
P(d_2 = \text{Car} | m = 3, p = 1) = 1 - P(d_1 = \text{Car} | m = 3, p = 1) - P(d_3 = \text{Car} | m = 3, p = 1) = 1 - \frac{1}{3} - 0 = \frac{2}{3}
\]

You should pick door 2!
Solution (2)

doors := [0,0,0];
prize := uniformInt(0,2);
doors[prize] = 1;
choice := uniformInt(0,2);
open := (0 + 1 + 2) - choice - prize;
if choice == prize
    open := (prize + uniformInt(1,2)) % 3;
return Expectation(doors[choice])<Expectation(doors[other]);