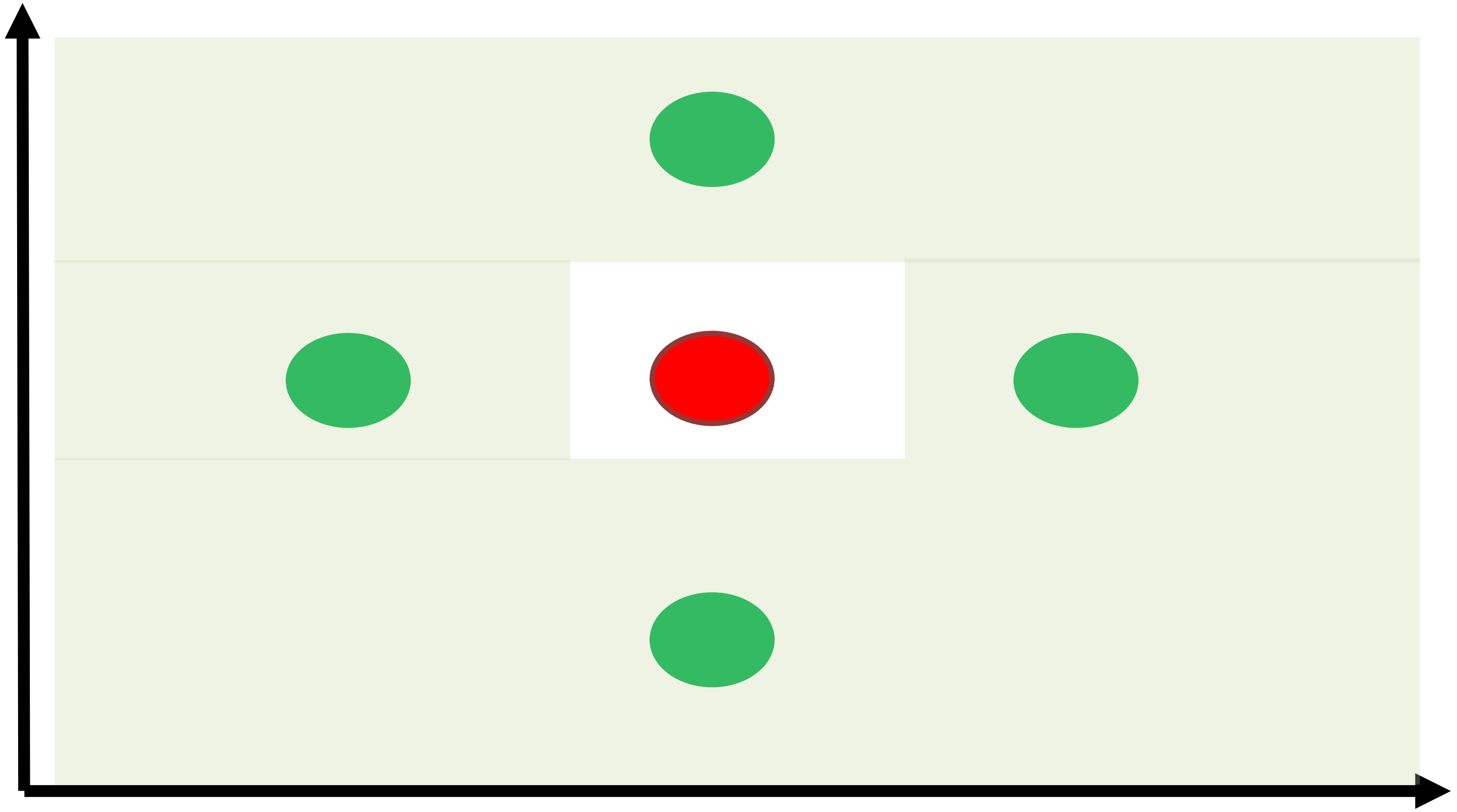
Deep Learning: Question

Given labelled data:

$$D = \{(1,3) \to 1\,, (3,1) \to 1, (3,5) \to 1, (5,3) \to 1, (3,3) \to 0\}$$

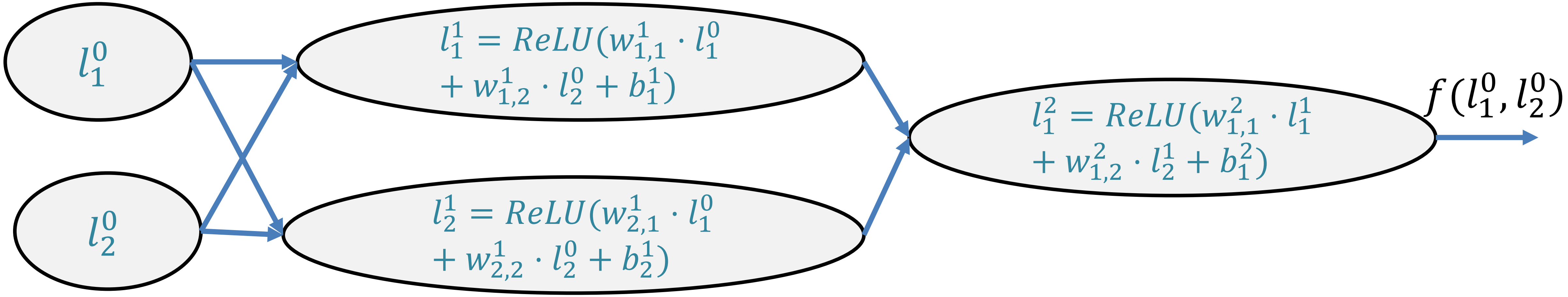


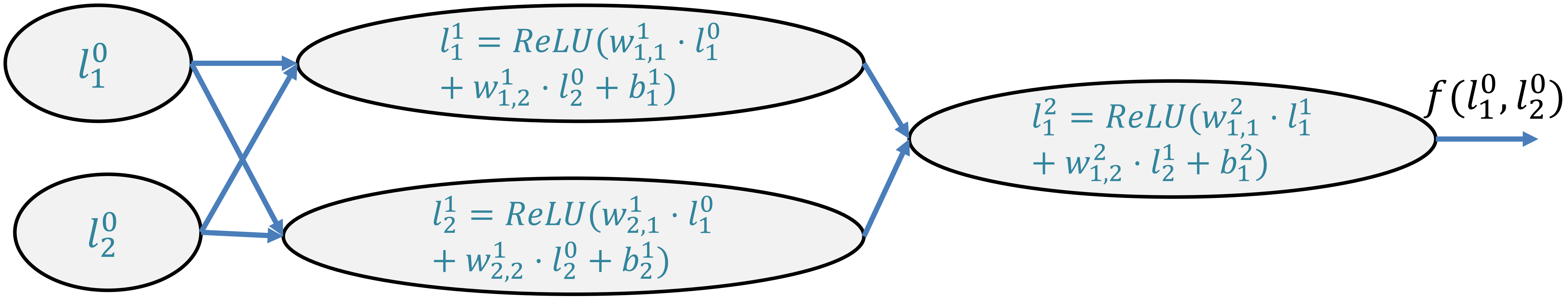
Deep Learning: Question

Given labelled data:

$$D = \{(1,3) \to 1\,, (3,1) \to 1, (3,5) \to 1, (5,3) \to 1, (3,3) \to 0\}$$

Consider a deep learning model with one hidden layer with two neurons.



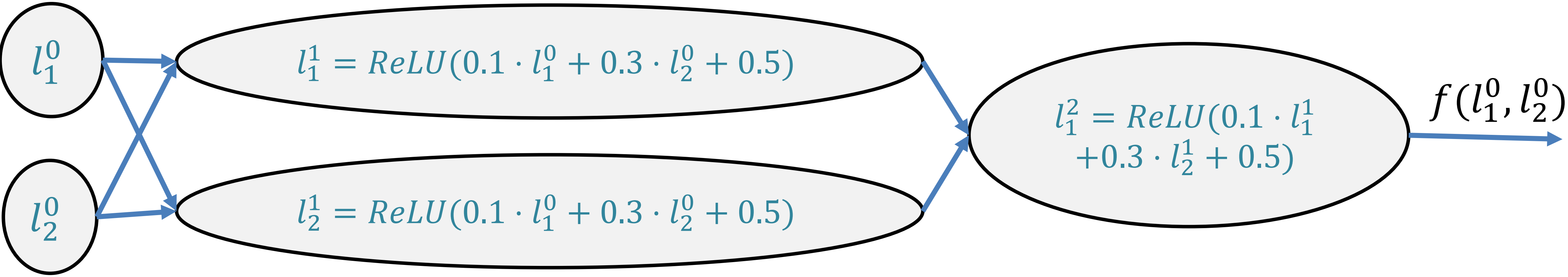


Define:

$$w_{1,1}^1 = w_{2,1}^1 = w_{1,1}^2 = 0.1$$
$$w_{1,2}^1 = w_{2,2}^1 = w_{1,2}^2 = 0.3$$
$$b_1^1 = b_2^1 = b_1^2 = 0.5$$

Compute a single iteration of backpropagation.

Initial Model



l_1^0	l_2^0	f^*	l_1^1	l_2^1	$l_1^2 = f(l_1^0, l_2^0)$
1	3	1	$ReLU(0.1 \cdot 1$	$ReLU(0.1 \cdot 1$	$ReLU(0.1 \cdot 1.5$ $+ 0.3 \cdot 1.5$
3	1	1	1.1	1.1	0.94
3	5	1	2.3	2.3	1.42
5	3	1	1.9	1.9	1.26
3	3	0	1.7	1.7	1.18

Initial MSE

$$\text{MSE} = \Sigma_{(l_1^0, l_2^0) \in D} \left(f^*(l_1^0, l_2^0) - f(l_1^0, l_2^0) \right)^2$$

$$(1 - 1.1)^2 + (1 - 0.94)^2 + (1 - 1.42)^2 + (1 - 1.26)^2 + (0 - 1.18)^2 = 1.65$$

Finding Minima via Gradient Descent

1. **Initialize** randomly with certain values for weights/bias a_0 , set $\mathrm{n} = 0$

2. Compute the **gradient** of the MSE at a_n

3. The **next point** is the one maximizing the decrease in MSE

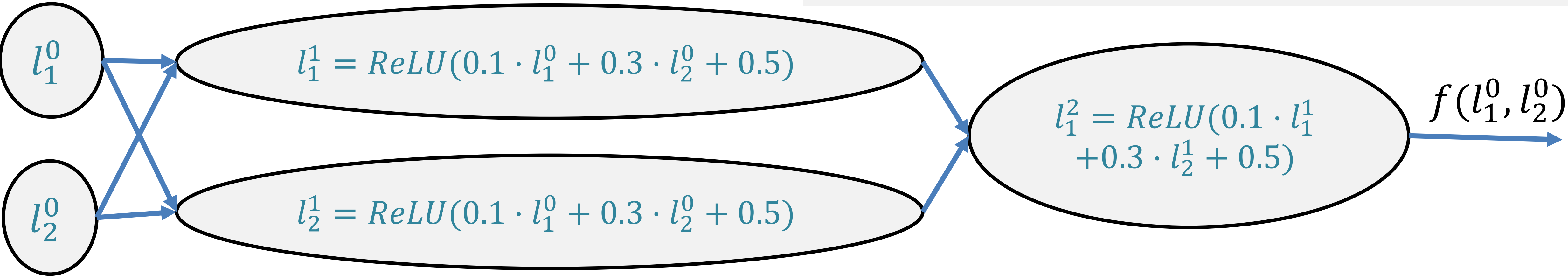
   $a_{n+1} = a_n - \gamma \nabla MSE(a_n)$ (γ is called the learning rate)

   n = n + 1

4. If **loss is small enough**, complete; otherwise, repeat 2

Gradients

$$\text{MSE} = \Sigma_{(l_1^0, l_2^0)\in D} \left(f^*(l_1^0, l_2^0) - f(l_1^0, l_2^0)\right)^2$$

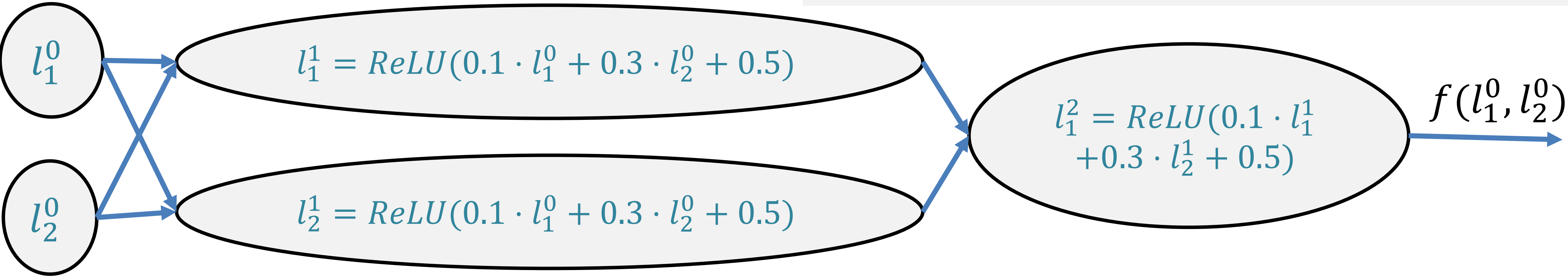


$$\frac{\partial MSE}{\partial w_{1,1}^2} = \frac{\partial MSE}{\partial l_1^2} \cdot \frac{\partial l_1^2}{\partial p_1^2} \cdot \frac{\partial p_1^2}{\partial w_{1,1}^2} = \Sigma_{(l_1^0, l_2^0)\in D_{Tr}} 2(f^*(l_1^0, l_2^0) - l_1^2) \cdot (-1) \cdot l_1^1$$

$$= -2[(1 - 1.1) \cdot 1.5 + (1 - 0.94) \cdot 1.1 + (1 - 1.42) \cdot 2.3 + (1 - 1.26)$$

Gradients

$$\text{MSE} = \Sigma_{(l_1^0, l_2^0) \in D} \left(f^*(l_1^0, l_2^0) - f(l_1^0, l_2^0) \right)^2$$

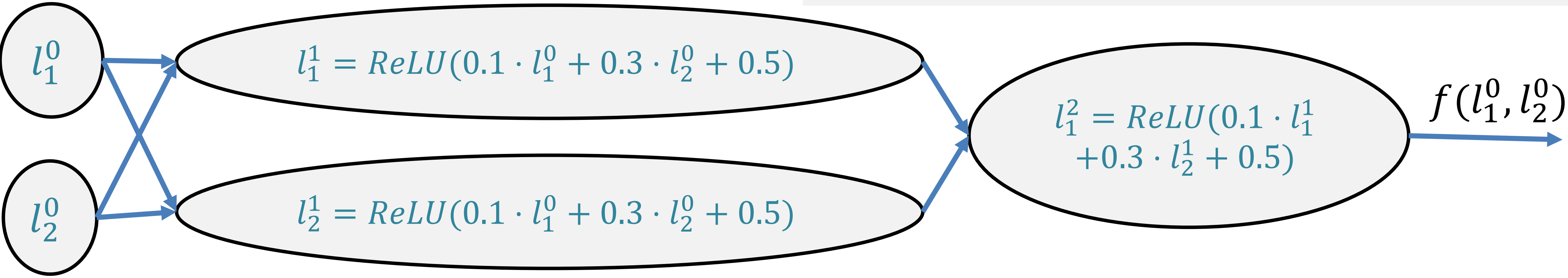


$$\frac{\partial MSE}{\partial w_{1,2}^2} = \frac{\partial MSE}{\partial l_1^2} \cdot \frac{\partial l_1^2}{\partial p_1^2} \cdot \frac{\partial p_1^2}{\partial w_{1,2}^2} = \Sigma_{(l_1^0, l_2^0) \in D_{Tr}} 2(f^*(l_1^0, l_2^0) - l_1^2) \cdot (-1) \cdot l_2^1$$

$$= -2[(1 - 1.1) \cdot 1.5 + (1 - 0.94) \cdot 1.1 + (1 - 1.42) \cdot 2.3 + (1 - 1.26)$$

Gradients

$$\text{MSE} = \Sigma_{(l_1^0, l_2^0) \in D} \left(f^*(l_1^0, l_2^0) - f(l_1^0, l_2^0) \right)^2$$



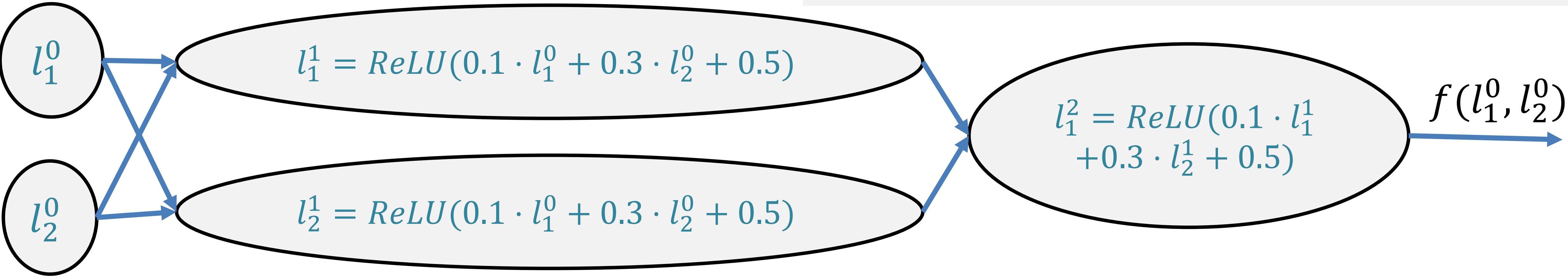
$$\frac{\partial MSE}{\partial b_1^2} = \frac{\partial MSE}{\partial l_1^2} \cdot \frac{\partial l_1^2}{\partial p_1^2} \cdot \frac{\partial p_1^2}{\partial b_1^2} = \Sigma_{(l_1^0, l_2^0) \in D_{Tr}} 2(f^*(l_1^0, l_2^0) - l_1^2) \cdot (-1) \cdot 1$$

$$= -2[(1 - 1.1) + (1 - 0.94) + (1 - 1.42) + (1 - 1.26) + (0 - 1.18)]$$

$$= 3.8$$

Gradients

$$\text{MSE} = \Sigma_{(l_1^0, l_2^0) \in D} \left(f^*(l_1^0, l_2^0) - f(l_1^0, l_2^0) \right)^2$$

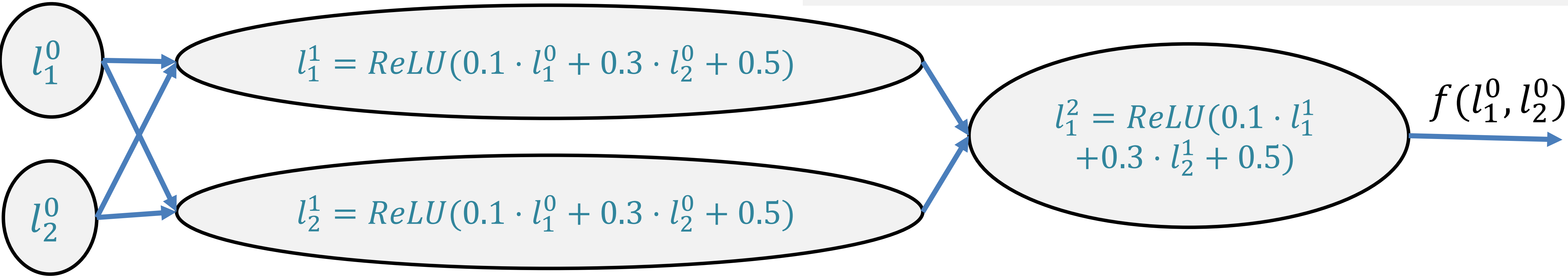


$$\frac{\partial MSE}{\partial w_{1,1}^1} = \frac{\partial MSE}{\partial l_1^2} \cdot \frac{\partial l_1^2}{\partial p_1^2} \cdot \frac{\partial p_1^2}{\partial l_1^1} \cdot \frac{\partial l_1^1}{\partial p_1^1} \cdot \frac{\partial p_1^1}{\partial w_{1,1}^1} = \frac{\partial MSE}{\partial l_1^2} \cdot \frac{\partial l_1^2}{\partial p_1^2} \Sigma_{(l_1^0, l_2^0) \in D_{Tr}} w_{1,1}^2 \cdot 1 \cdot l_1^0$$

$$= 3.8 \cdot 0.1 \cdot [1 + 3 + 3 + 3 + 5] = 15.38$$

Gradients

$$\text{MSE} = \Sigma_{(l_1^0, l_2^0)\in D}\left(f^*(l_1^0, l_2^0) - f(l_1^0, l_2^0)\right)^2$$



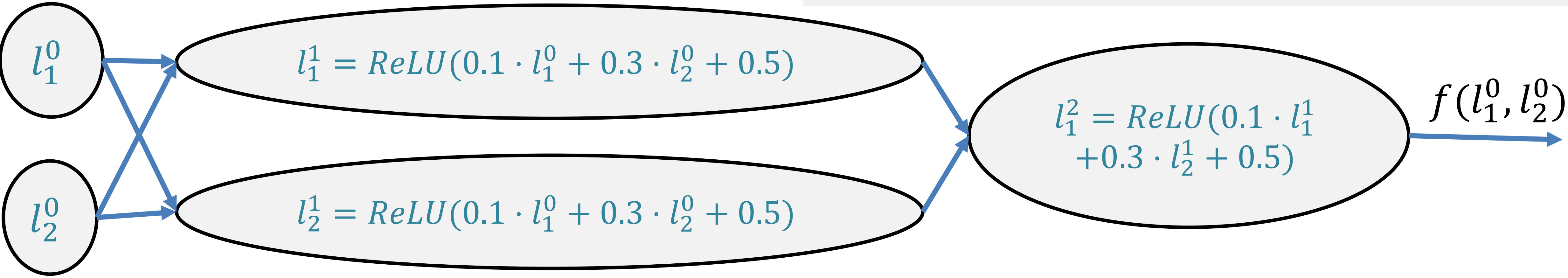
$$\frac{\partial MSE}{\partial w_{1,2}^1} = \frac{\partial MSE}{\partial l_1^2}\cdot\frac{\partial l_1^2}{\partial p_1^2}\cdot\frac{\partial p_1^2}{\partial l_1^1}\cdot\frac{\partial l_1^1}{\partial p_1^1}\cdot\frac{\partial p_1^1}{\partial w_{1,1}^1} = \frac{\partial MSE}{\partial l_1^2}\cdot\frac{\partial l_1^2}{\partial p_1^2}\Sigma_{(l_1^0, l_2^0)\in D_{Tr}} w_{1,1}^2 \cdot 1 \cdot l_2^0$$

$$= 3.8 \cdot 0.1 \cdot [1 + 3 + 3 + 3 + 5] = 15.38$$

$$\frac{\partial MSE}{\partial b_1^1} = \frac{\partial MSE}{\partial l_1^2}\cdot\frac{\partial l_1^2}{\partial p_1^2}\cdot\frac{\partial p_1^2}{\partial l_1^1}\cdot\frac{\partial l_1^1}{\partial p_1^1}\cdot\frac{\partial p_1^1}{\partial w_{1,1}^1} = \frac{\partial MSE}{\partial l_1^2}\cdot\frac{\partial l_1^2}{\partial p_1^2}\Sigma_{(l_1^0, l_2^0)\in D_{Tr}} w_{1,1}^2 \cdot 1 = 3.8 \cdot 0.1 = 0.38$$

Gradients

$$\text{MSE} = \Sigma_{(l_1^0,l_2^0)\in D}\left(f^*(l_1^0,l_2^0) - f(l_1^0,l_2^0)\right)^2$$



$$\frac{\partial MSE}{\partial w_{2,1}^1} = \frac{\partial MSE}{\partial l_1^2} \cdot \frac{\partial l_1^2}{\partial p_1^2} \cdot \frac{\partial p_1^2}{\partial l_1^1} \cdot \frac{\partial l_1^1}{\partial p_1^1} \cdot \frac{\partial p_1^1}{\partial w_{1,1}^1} = \frac{\partial MSE}{\partial l_1^2} \cdot \frac{\partial l_1^2}{\partial p_1^2} \Sigma_{(l_1^0,l_2^0)\in D_{Tr}} w_{2,1}^2 \cdot 1 \cdot l_1^0$$

$$= 3.8 \cdot 0.3 \cdot [1 + 3 + 3 + 3 + 5] = 17.1$$

$$\frac{\partial MSE}{\partial b_2^1} = \frac{\partial MSE}{\partial l_1^2} \cdot \frac{\partial l_1^2}{\partial p_1^2} \cdot \frac{\partial p_1^2}{\partial l_1^1} \cdot \frac{\partial l_1^1}{\partial p_1^1} \cdot \frac{\partial p_1^1}{\partial w_{1,1}^1} = \frac{\partial MSE}{\partial l_1^2} \cdot \frac{\partial l_1^2}{\partial p_1^2} \Sigma_{(l_1^0,l_2^0)\in D_{Tr}} w_{2,1}^2 \cdot 1 = 3.8 \cdot 0.3 = 1.14$$

Finding Minima via Gradient Descent

The **next point** is the one maximizing the decrease in MSE

$$a_{n+1} = a_n - \gamma \nabla MSE(a_n) \qquad (\gamma \text{ is called the learning rate})$$

Pick $\gamma = 0.01$

$w_{1,1}^2 = 0.1 - 0.01 \cdot 7.1 = 0.029$ $\quad$ $b_1^2 = 0.5 - 0.01 \cdot 3.8 = 0.462$

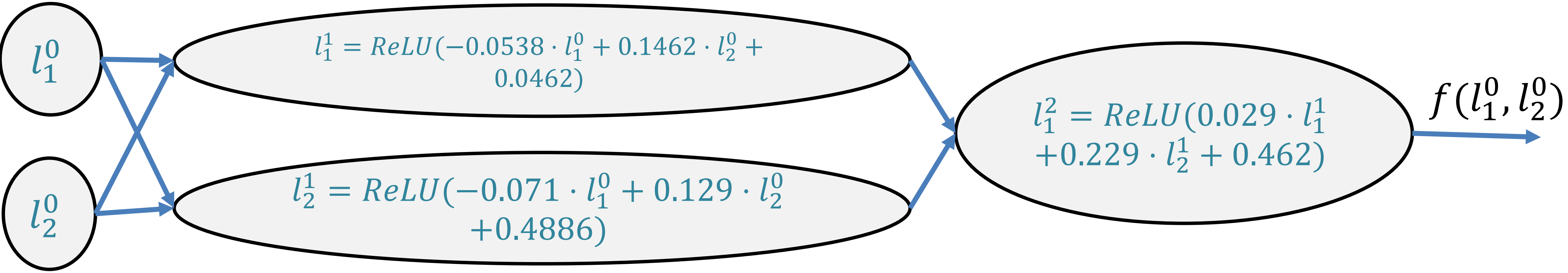
$w_{1,2}^2 = 0.3 - 0.01 \cdot 7.1 = 0.229$ $\quad$ $w_{1,1}^1 = 0.1 - 0.01 \cdot 15.38 = -0.0538$

$w_{1,2}^1 = 0.3 - 0.01 \cdot 15.38 = 0.1462$ $\quad$ $b_1^1 = 0.5 - 0.01 \cdot 0.38 = 0.0462$

$w_{2,1}^1 = 0.1 - 0.01 \cdot 17.1 = -0.071$ $\quad$ $w_{2,2}^1 = 0.3 - 0.01 \cdot 17.1 = 0.129$

$b_2^1 = 0.5 - 0.01 \cdot 1.14 = 0.4886$

Updated Model



l_1^0	l_2^0	f^*	l_1^1	l_2^1	$l_1^2 = f(l_1^0, l_2^0)$
1	3	1	0.431	0.8046	0.6589
3	1	1	0.031	0.4046	0.5556
3	5	1	0.6158	0.9206	0.6907
5	3	1	0.2158	0.5206	0.5875
3	3	0	0.3234	0.6626	0.6231

MSE

$$\text{MSE} = \Sigma_{(l_1^0\,, l_2^0) \in D} \left(f^*(l_1^0\,, l_2^0) - f(l_1^0\,, l_2^0) \right)^2$$

$$(1 - 0.6589)^2 + (1 - 0.5556)^2 + (1 - 0.6907)^2 + (1 - 0.5875)^2$$
$$+ (0 - 0.6231)^2 = 0.9679$$