The Rise and Fall of Linear Temporal Logic

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Monadic Logic

**Monadic Class**: First-order logic with $=$ and monadic predicates – captures *syllogisms*.

- $(\forall x)P(x), (\forall x)(P(x) \rightarrow Q(x)) \models (\forall x)Q(x)$

[Löwenheim, 1915]: The Monadic Class is decidable.

- **Proof**: Bounded-model property – if a sentence is satisfiable, it is satisfiable in a structure of bounded size.
- **Proof technique**: quantifier elimination.

**Monadic Second-Order Logic**: Allow second-order quantification on monadic predicates.

[Skolem, 1919]: Monadic Second-Order Logic is decidable – via bounded-model property and quantifier elimination.

**Question**: What about $<$?
Nondeterministic Finite Automata

\[ A = (\Sigma, S, S_0, \rho, F) \]

- **Alphabet**: \( \Sigma \)
- **States**: \( S \)
- **Initial states**: \( S_0 \subseteq S \)
- **Nondeterministic transition function**: 
  \( \rho : S \times \Sigma \rightarrow 2^S \)
- **Accepting states**: \( F \subseteq S \)

**Input word**: \( a_0, a_1, \ldots, a_{n-1} \)

**Run**: \( s_0, s_1, \ldots, s_n \)
- \( s_0 \in S_0 \)
- \( s_{i+1} \in \rho(s_i, a_i) \) for \( i \geq 0 \)

**Acceptance**: \( s_n \in F \)

**Recognition**: \( L(A) \) – words accepted by \( A \).

**Example**: 
\[
\begin{array}{c}
\bullet & \xrightarrow{1} & \bullet \\
\uparrow & & \downarrow \\
0 & & 1 \\
\end{array}
\]

– ends with 1’s

**Fact**: NFAs define the class \( \text{Reg} \) of regular languages.
Logic of Finite Words

View finite word $w = a_0, \ldots, a_{n-1}$ over alphabet $\Sigma$ as a mathematical structure:
- Domain: $0, \ldots, n - 1$
- Binary relation: $<$
- Unary relations: $\{P_a : a \in \Sigma\}$

**First-Order Logic (FO):**
- Unary atomic formulas: $P_a(x)$ ($a \in \Sigma$)
- Binary atomic formulas: $x < y$

**Example:** $(\exists x)((\forall y)(\neg(x < y)) \land P_a(x))$ — last letter is $a$.

**Monadic Second-Order Logic (MSO):**
- Monadic second-order quantifier: $\exists Q$
- New unary atomic formulas: $Q(x)$
NFA vs. MSO

**Theorem** [Büchi, Elgot, Trakhtenbrot, 1957-8 (independently)]: MSO \( \equiv \) NFA

- Both MSO and NFA define the class Reg.

**Proof**: Effective

- From NFA to MSO \((A \mapsto \varphi_A)\)
  - Existence of run – existential monadic quantification
  - Proper transitions and acceptance - first-order formula

- From MSO to NFA \((\varphi \mapsto A_\varphi)\): closure of NFAs under
  - **Union** – disjunction
  - **Projection** – existential quantification
  - **Complementation** – negation
NFA Nonemptiness

**Nonemptiness**: \( L(A) \neq \emptyset \)

**Nonemptiness Problem**: Decide if given \( A \) is nonempty.

**Directed Graph** \( G_A = (S, E) \) of NFA \( A = (\Sigma, S, S_0, \rho, F) \):
- **Nodes**: \( S \)
- **Edges**: \( E = \{(s, t) : t \in \rho(s, a) \text{ for some } a \in \Sigma\} \)

**Lemma**: \( A \) is nonempty iff there is a path in \( G_A \) from \( S_0 \) to \( F \).

- Decidable in time linear in size of \( A \), using *breadth-first search* or *depth-first search*. 
MSO Satisfiability – Finite Words

**Satisfiability**: $\text{models}(\psi) \neq \emptyset$

**Satisfiability Problem**: Decide if given $\psi$ is satisfiable.

**Lemma**: $\psi$ is satisfiable iff $A_\psi$ is nonempty.

**Corollary**: MSO satisfiability is decidable.

- Translate $\psi$ to $A_\psi$.
- Check nonemptiness of $A_\psi$.

**Complexity**:

- **Upper Bound**: Nonelementary Growth
  \[2 \cdot 2^n\]
  (tower of height $O(n)$)

- **Lower Bound** [Stockmeyer, 1974]: Satisfiability of FO over finite words is nonelementary (no bounded-height tower).
Sequential Circuits

Church, 1957: Use logic to specify sequential circuits.

**Sequential circuits:** \( C = (I, O, R, f, g, R_0) \)
- \( I \): input signals
- \( O \): output signals
- \( R \): sequential elements
- \( f : 2^I \times 2^R \to 2^R \): transition function
- \( g : 2^R \to 2^O \): output function
- \( R_0 \in 2^R \): initial assignment

**Trace:** element of \((2^I \times 2^R \times 2^O)^\omega\)
\( t = (I_0, R_0, O_0), (I_1, R_1, O_1), \ldots \)
- \( R_{j+1} = f(I_j, R_j) \)
- \( O_j = g(R_j) \)
Specifying Traces

View infinite trace \( t = (I_0, R_0, O_0), (I_1, R_1, O_1), \ldots \) as a mathematical structure:

- Domain: \( N \)
- Binary relation: \(<\)
- Unary relations: \( I \cup R \cup O \)

First-Order Logic (FO):

- Unary atomic formulas: \( P(x) \ (P \in I \cup R \cup O) \)
- Binary atomic formulas: \( x < y \)

Example: \( (\forall x)(\exists y)(x < y \land P(y)) \) – \( P \) holds i.o.

Monadic Second-Order Logic (MSO):

- Monadic second-order quantifier: \( \exists Q \)
- New unary atomic formulas: \( Q(x) \)

Model-Checking Problem: Given circuit \( C \) and formula \( \varphi \); does \( \varphi \) hold in all traces of \( C \)?

Easy Observation: Model-checking problem reducible to satisfiability problem – use FO to encode the “logic” (i.e., \( f, g \)) of the circuit \( C \).
Büchi Automata

Büchi Automaton: \( A = (\Sigma, S, S_0, \rho, F) \)

- **Alphabet:** \( \Sigma \)
- **States:** \( S \)
- **Initial states:** \( S_0 \subseteq S \)
- **Transition function:** \( \rho : S \times \Sigma \rightarrow 2^S \)
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**Input word:** \( a_0, a_1, \ldots \)

**Run:** \( s_0, s_1, \ldots \)

- \( s_0 \in S_0 \)
- \( s_{i+1} \in \rho(s_i, a_i) \) for \( i \geq 0 \)

**Acceptance:** \( F \) visited infinitely often

- infinitely many 1’s

**Fact:** Büchi automata define the class \( \omega-\text{Reg} \) of \( \omega \)-regular languages.
Logic vs. Automata II

**Paradigm**: Compile high-level logical specifications into low-level finite-state language

**Compilation Theorem**: [Büchi, 1960] Given an MSO formula \( \varphi \), one can construct a Büchi automaton \( A_\varphi \) such that a trace \( \sigma \) satisfies \( \varphi \) if and only if \( \sigma \) is accepted by \( A_\varphi \).

**MSO Satisfiability Algorithm**:

1. \( \varphi \) is satisfiable iff \( L(A_\varphi) \neq \emptyset \)
2. \( L(\Sigma, S, S_0, \rho, F) \neq \emptyset \) iff there is a path from \( S_0 \) to a state \( f \in F \) and a cycle from \( f \) to itself.

**Corollary** [Church, 1960]: Model checking sequential circuits wrt MSO specs is decidable.

Church, 1960: “Algorithm not very efficient” (\emph{nonelementary complexity}, [Stockmeyer, 1974]).
Temporal Logic

Prior, 1914–1969, Philosophical Preoccupations:

- **Religion**: Methodist, Presbytarian, atheist, agnostic
- **Ethics**: “Logic and The Basis of Ethics”, 1949
- **Free Will, Predestination, and Foreknowledge**:
  - “The future is to some extent, even if it is only a very small extent, something we can make for ourselves”.
  - “Of what will be, it has now been the case that it will be.”
  - “There is a deity who infallibly knows the entire future.”

Mary Prior: “I remember his waking me one night [in 1953], coming and sitting on my bed, . . ., and saying he thought one could make a formalised tense logic.”

- 1957: “Time and Modality”
Linear vs. Branching Time, A

- Prior’s first lecture on tense logic, Wellington University, 1954: linear time.
- Prior’s “Time and modality”, 1957: relationship between linear tense logic and modal logic.
- Sep. 1958, letter from Saul Kripke: “[I]n an indetermined system, we perhaps should not regard time as a linear series, as you have done. Given the present moment, there are several possibilities for what the next moment may be like – and for each possible next moment, there are several possibilities for the moment after that. Thus the situation takes the form, not of a linear sequence, but of a ‘tree’.” (Kripke was a high-school student, not quite 18, in Omaha, Nebraska.)
Linear vs. Branching Time, B

- **Linear time**: a system induces a set of traces

- **Specs**: describe traces

  ________________________________ . . .

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  ________________________________ . . .

- **Branching time**: a system induces a trace tree

- **Specs**: describe trace trees

![Trace Tree Diagram]
Linear vs. Branching Time, C

- Prior developed the idea into Ockhamist and Peircean theories of branching time (branching-time logic *without* path quantifiers)

Sample formula: $C \overline{K} M p \overline{M} q A \overline{M} K p M q M K q M p$

- Burgess, 1978: “Prior would agree that the determinist sees time as a line and the indeterminist sees times as a system of forking paths.”
Linear vs. Branching Time, D

Philosophical Conundrum

- Prior:
  - Nature of course of time – branching
  - Nature of course of events – linear

- Rescher:
  - Nature of time – linear
  - Nature of course of events – branching
  - “We have ‘branching in time’, not ‘branching of time’”.

Linear time: Hans Kamp, Dana Scott and others continued the development of linear time during the 1960s.
Temporal and Classical Logics

Key Theorem:

- Kamp, 1968: Linear temporal logic with past and binary temporal connectives (“until” and “since”), over the integers, has precisely the expressive power of FO.
The Temporal Logic of Programs

Precursors:

- Prior: “There are practical gains to be had from this study too, for example in the representation of time-delay in computer circuits”

- Rescher & Urquhart, 1971: applications to processes (“a programmed sequence of states, deterministic or stochastic”)

“Big Bang 1” [Pnueli, 1977]:
- Future linear temporal logic (LTL) as a logic for the specification of non-terminating programs
- Temporal logic with “eventually” and “always” (later, with “next” and “until”)
- Model checking via reduction to MSO and automata

Crux: Need to specify ongoing behavior rather than input/output relation!
Linear Temporal Logic

Linear Temporal logic (LTL): logic of temporal sequences (Pnueli, 1977)

Main feature: time is implicit

- **next** $\varphi$: $\varphi$ holds in the next state.
- **eventually** $\varphi$: $\varphi$ holds eventually
- **always** $\varphi$: $\varphi$ holds from now on
- **$\varphi$ until** $\psi$: $\varphi$ holds until $\psi$ holds.

\[ \pi, w \models \text{next } \varphi \text{ if } w \bullet \ldots \bullet \varphi \bullet \ldots \\]

\[ \pi, w \models \varphi \text{ until } \psi \text{ if } w \bullet \varphi \bullet \varphi \bullet \varphi \psi \bullet \ldots \]
Examples

Psalm 34:14: “Depart from evil and do good”

- always not \((\text{CS}_1 \text{ and } \text{CS}_2)\): mutual exclusion (safety)

- always (Request implies eventually Grant): liveness

- always (Request implies (Request until Grant)): liveness
Expressive Power

- Gabbay, Pnueli, Shelah & Stavi, 1980: Propositional LTL over the naturals has precisely the expressive power of FO.
- Thomas, 1979: FO over naturals has the expressive power of star-free $\omega$-regular expressions

Summary: \( \text{LTL} = \text{FO} = \text{star-free } \omega\text{-RE} < \text{MSO} = \omega\text{-RE} \)

Meyer on LTL, 1980, in “Ten Thousand and One Logics of Programming”:

“The corollary due to Meyer – I have to get in my controversial remark – is that that [GPSS’80] makes it theoretically uninteresting.”
Recall: Satisfiability of FO over traces is non-elementary

Contrast with LTL:
- Wolper, 1981: LTL satisfiability is in EXPTIME.

Basic Technique: tableau (influenced by branching-time techniques)
PLTL

Lichtenstein, Pnueli, & Zuck, 1985: past-time connectives are useful in LTL:

- yesterday $q$: $q$ was true in the previous state
- past $p$: $q$ was true sometime in the past
- $p$ since $q$: $p$ has been true since $q$ was true

**Example:** always ($rcv \rightarrow \text{past snt}$)

**Theorem**
- Expressively equivalent to LTL [LPZ'85]
- Satisfiability of PLTL is PSPACE-complete [LPZ'85]
- PLTL is exponentially more succinct than LTL [Markey, 2002]
"Big Bang 2" [Clarke & Emerson, 1981, Queille & Sifakis, 1982]: Model checking programs of size $m$ wrt CTL formulas of size $n$ can be done in time $mn$.

Linear-Time Response [Lichtenstein & Pnueli, 1985]: Model checking programs of size $m$ wrt LTL formulas of size $n$ can be done in time $m2^{O(n)}$ (tableau-based).

Seemingly:

- *Automata*: Nonelementary
- *Tableaux*: exponential
Exponential-Compilation Theorem:

[V. & Wolper, 1983–1986]

Given an LTL formula \( \varphi \) of size \( n \), one can construct a Büchi automaton \( A_\varphi \) of size \( 2^{O(n)} \) such that a trace \( \sigma \) satisfies \( \varphi \) if and only if \( \sigma \) is accepted by \( A_\varphi \).

**Automata-Theoretic Algorithms:**

1. **LTL Satisfiability:**
   \( \varphi \) is satisfiable iff \( L(A_\varphi) \neq \emptyset \) (PSPACE)

2. **LTL Model Checking:**
   \( M \models \varphi \) iff \( L(M \times A_{\neg \varphi}) = \emptyset \) (\( m2^{O(n)} \))

Vardi, 1988: Also with past.
Reduction to Practice

**Practical Theory:**

- Courcoubetis, V., Yannakakis & Wolper, 1989: Optimized search algorithm for explicit model checking
- Burch, Clarke, McMillan, Dill & Hwang, 1990: Symbolic algorithm for LTL compilation
- Clarke, Grumberg & Hamaguchi, 1994: Optimized symbolic algorithm for LTL compilation
- Gerth, Peled, V. & Wolper, 1995: Optimized explicit algorithm for LTL compilation

**Implementation:**

- Spin [Holzmann, 1995]: Promela w. LTL:
- SMV [McMillan, 1995]: SMV w. LTL

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*Satisfactory solution to Church’s problem? Almost, but not quite, since LTL<MSO=ω-RE.*
Enhancing Expressiveness

- Wolper, 1981: Enhance LTL with grammar operators, retaining EXPTIME-ness (PSPACE [SC’82])
- V. & Wolper, 1983: Enhance LTL with automata, retaining PSPACE-completeness
- Sistla, V. & Wolper, 1985: Enhance LTL with 2nd-order quantification, losing elementariness
- V., 1989: Enhance LTL with fixpoints (as in Kozen’s $\mu$-calculus), retaining PSPACE-completeness

**Bottom Line:** ETL (LTL w. automata) = $\mu$TL (LTL w. fixpoints) = MSO, and has exponential- compilation property.
Dynamic and Branching-Time Logics

**Dynamic Logic** [Pratt, 1976]:
- The □φ of modal logic can be taken to mean “φ holds after an execution of a program step”.
- Dynamic modalities:
  - [α]φ – φ holds after all executions of α.
  - ψ → [α]φ corresponds to Hoare triple {ψ}α{φ}.

**Propositional Dynamic Logic** [Fischer & Ladner, 1977]: Boolean propositions, programs – regular expressions over atomic programs.

**Satisfiability** [Pratt, 1978]: EXPTIME – using tableau-based algorithm

Branching-Time Logic

From dynamic logic back to temporal logic:
The dynamic-logic view is clearly branching; what is the analog for temporal logic?

- Emerson & Clarke, 1980: correctness properties as fixpoints over computation trees
- Ben-Ari, Manna & Pnueli, 1981: branching-time logic UB; satisfiability in EXPTIME using tableau
- Clarke & Emerson, 1981: branching-time logic CTL; efficient model checking
- Emerson & Halpern, 1983: branching-time logic CTL* – ultimate branching-time logic

Key Idea: Prior missed path quantifiers
- $\forall$ eventually $p$: on all possible futures, $p$ eventually happen.
Linear vs. Branching Temporal Logics

- **Linear time**: a system generates a set of computations
- **Specs**: describe computations
- **LTL**: \( \text{always} (\text{request} \rightarrow \text{eventually grant}) \)

- **Branching time**: a system generates a computation tree
- **Specs**: describe computation trees
- **CTL**: \( \forall \text{always} (\text{request} \rightarrow \forall \text{eventually grant}) \)
Combining Dynamic and Temporal Logics

Two distinct perspectives:
- Temporal logic: *state based*
- Dynamic logic: *action based*

Symbiosis:
- Harel, Kozen & Parikh, 1980: Process Logic (branching time)
- V. & Wolper, 1983: Yet Another Process Logic (branching time)
- Harel and Peleg, 1985: Regular Process Logic (linear time)
- Henriksen and Thiagarajan, 1997: Dynamic LTL (linear time)

Tech Transfer:
- Beer, Ben-David & Landver, IBM, 1998: RCTL (branching time)
- Beer, Ben-David, Eisner, Fisman, Gringauze, Rodeh, IBM, 2001: Sugar (branching time)
From LTL to PSL

Model Checking at Intel

*Prehistory:*

- 1990: successful feasibility study using Kurshan’s COSPAN
- 1992: a pilot project using CMU’s SMV
- 1995: an internally developed (linear time) property-specification language

*History:*

- 1997: Development of 2nd-generation technology started (engine and language)
- 1999: BDD-based model checker released
- 2000: SAT-based model checker released
- 2000: *ForSpec* (language) released
Dr. Vardi Goes to Intel

1997: (w. Fix, Hadash, Kesten, & Sananes)
V.: How about LTL?
F., H., K., & S.: Not expressive enough.

V.: How about ETL? $\mu$TL?
F., H., K., & S.: Users will object.

1998 (w. Landver)

V.: How about ETL?
L.: Users will object.
L.: How about regular expressions?
V.: They are equivalent to automata!

**RELTL:** LTL plus dynamic modalities, interpreted linearly – $[e] \varphi$
E.g.: $[true^*, \text{send, !cancel}]\text{sent}$

**Easy:** RELTL=ETL=$\omega$-RE

**ForSpec:** RELTL + hardware features (clocks and resets) [Armoni, Fix, Flaisher, Gerth, Ginsburg, Kanza, Landver, Mador-Haim, Singerman, Tiemeyer, V., Zbar]
From ForSpec to PSL

**Industrial Standardization:**
- Process started in 2000
- Four candidates: IBM’s Sugar, Intel’s ForSpec, Motorola’s CBV, and Verisity’s E.
- Fierce debate on linear vs. branching time

**Outcome:**
- Big political win for IBM (see references to PSL/Sugar)
- Big technical win for Intel
  - PSL is LTL + RE + clocks + resets
  - Branching-time extension as an acknowledgement to Sugar
  - Some evolution over time in hardware features
- Major influence on the design of SVA (another industrial standard)

**Bottom Line:** Huge push for model checking in industry.
What about the Past?

- Avoided in industrial languages due to implementation challenges
- Less important in model checking; if past events are important, then program would keep track of them.
- But, the past is important in specification (LPZ’85)!

Dax, Klaedtke, &Lange, 2010 (cf., Leucker & Sánchez, 2010): Regular Temporal Logic (RTL)
- PLTL
- Dynamic modalities: $[e]\varphi$
- Past Dynamic modalities: $[e]^{-}\varphi$

Theorem [DKL’10]
- Expressively equivalent to RELTL.
- Exponentially more succinct that RELTL.
- Satisfiability is PSPACE-complete
Linear Dynamic Logic (LDL)

Observations:

- Dynamic modalities subsume temporal connectives, e.g., \( \text{always } q \) is equivalent to \([\text{true}^*]q\)
- To capture past, add \textit{reverse} operator to REs.
  - \( a \): “consume” \( a \) and move forward.
  - \( a^- \): “consume” \( a \) and move backward.

Inspiration:

- PDL+converse [Pratt, 1976]
- Two-way navigation in XPath

Example: \([\text{true}^*, \text{rcv}]\langle(\text{true}^-)*\rangle\text{sent}\)

Theorem:

- Expressively equivalent to RELTL.
- Exponentially more succinct than RELTL.
- Satisfiability is PSPACE-complete.
LTL is Dead, Long Live LDL!

What was important about PLTL?
- Linear time
- Simple syntax
- Exponential-compilation property
- *Equivalence to FO*

What is important about LDL?
- Linear time
- Extremely simply syntax: REs (with *reverse*) and dynamic modalities
- Exponential-compilation property
- *Equivalence to MSO*

**Also:** easy to pronounce :-(